Multi-sorted relational clones on a two-element set

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Joint work with **Dmitriy Zhuk**

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$\to$ important for classifying algebras, studying "symmetries" of problems, ...

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Can we describe all the clones on $\{0, 1\}$?

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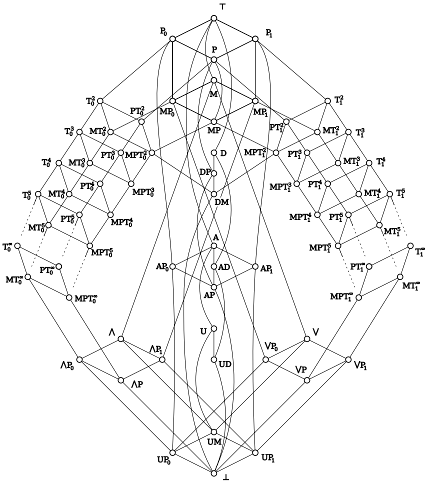
$\rightarrow$ Yes; E. Post, 1941, [1, 2]

Post's Lattice [1, 2]



Hasse diagram of all clones on $\{0, 1\}$
E. Post, 1941, [1, 2] (visualization [5])

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Theorem (Janov & Muchnik, 1959, [3]; Hulanicki & Swierczkowski, 1960, [4])

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$\rightarrow$ Complicated structure, only small parts described

Generalization – Multi-sorted Clones

Definition (multi-sorted clone)

Let $k \in \mathbb{N}$ and let A be a finite nonempty set. A set of k-tuples of operations on A (each operation in the tuple has the same arity), closed under forming (coordinate-wise) term operations is called a k-**clone** on A.

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example – 3-operation on $\{0, 1\}$

$$\begin{pmatrix} x^{(1)}, y^{(1)} \\ x^{(2)}, y^{(2)} \\ x^{(3)}, y^{(3)} \end{pmatrix} \mapsto \begin{pmatrix} x^{(1)} \wedge y^{(1)} \\ x^{(2)} \vee y^{(2)} \\ x^{(3)} \end{pmatrix}$$

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Describing Multi-sorted Clones on $\{0, 1\}$

Theorem (V. Taimanov, 1983–2022, [6–10])

Let $k \in \mathbb{N}$. There are countably many k-clones on $\{0, 1\}$. Each of them is generated by a finite set of k-operations.

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Other notable results

Maximal clones & generalizations (Taimanov, Romov, Marchenkov, 1983–1994, [6, 7, 11–14]), partial classification up to minion homomorphism (Barto & Kapytka, 2025, [15])

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$$\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_k \in \rho \implies f(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_k) \in \rho\,,$$

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$$\rho = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

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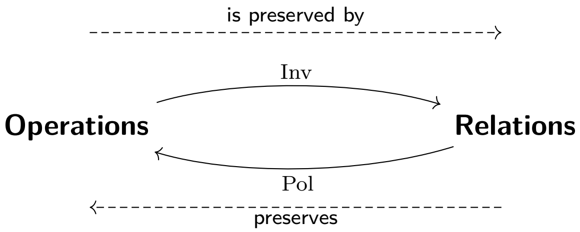
$$\rho = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \qquad \text{not preserved by } \vee \text{ since } \begin{pmatrix} 1 \vee 0 \\ 0 \vee 0 \\ 0 \vee 1 \end{pmatrix} \notin \rho$$

Inv-Pol Galois Connection I



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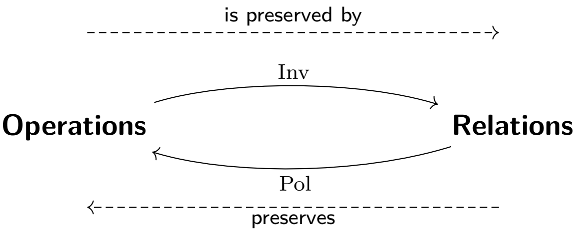

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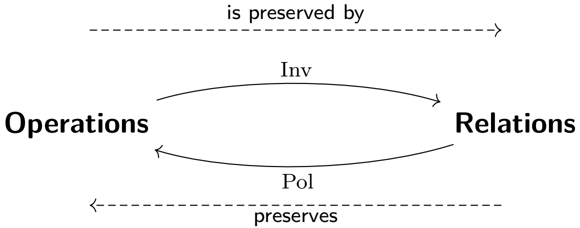

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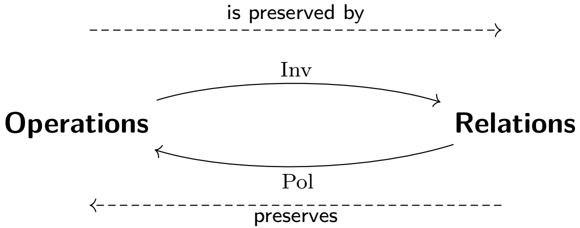

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– closed under pp formulas ($\exists$, $\wedge$, $=$)

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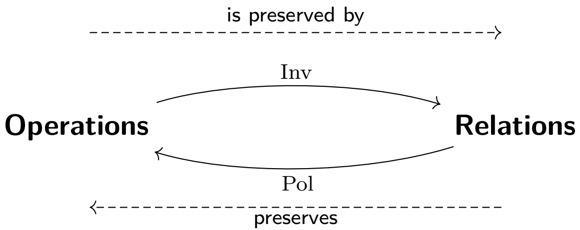

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pp formula: $\exists x_1 \dots \exists x_n \, \rho_1(\dots) \wedge \dots \wedge \rho_s(\dots)$

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A generalization holds for multi-sorted clones and **multi-sorted relational clones** $\rightarrow$ variables of relations have different sorts and only variables of the same sorts can be identified in pp formulas (Romov, 1973, [19]).

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$\exists u \exists v\, \rho(u, v, y) \wedge \tau(x, u)$ ... valid multi-sorted pp formula

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Example

If $\rho(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2 = 1) \vee (x_3 + x_4 + x_5 = 0)$, then, for example, $(1, 0, 1, 1, 1) \in \rho$ and $(1, 1, 1, 1, 1) \notin \rho$.

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Theorem (D. Zhuk, 2017, [20])

Let $k \in \mathbb{N}$ and let R be a k-sorted relational clone on $\{0, 1\}$. Every relation in R may be expressed as a conjunction of relations from $R \cap KR^k$.

Inv-sPol Galois Connection

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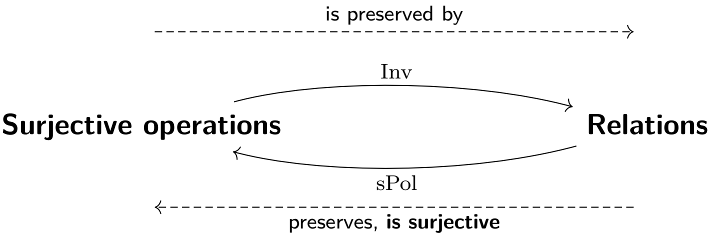

Closed sets (Börner & Bulatov & Chen & Jeavons & Krokhin, 2009, [21]):

Surjective clones
- sets of **surjective operations**
- closed under surj. term operations

Quantified relational clones
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(1) *Appending* or *removing* a dummy variable.

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$$\rho(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}) = \exists z\, \rho_1(z, x_1, \ldots, x_{n-1}) \wedge \rho_2(z, y_1, \ldots, y_{m-1}) \,.$$

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Theorem

Elementary operations are equivalent to general qpp formulas on $\{0, 1\}$.

Where are we?

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Key relations
Relations represented as disjunction of linear equations

All (q)pp formulas

Elementary operations
Restricted class of formulas with “simple” behaviour

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Canonical relations
Smallest class of relations sufficient to describe all quantified relational clones

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Canonical Relations

We denote the sorts of variables by superscript.

Definition (canonical relations)

A k-sorted relation without dummy variables is called *canonical* if, up to an order of its variables, it may be expressed in one of the following ways:

(c1) $x^i = 0 \vee y^i = 1$ for $i \in \{1, \ldots, k\}$,

(c2) $x^i = y^i \vee u^j = b$ for $i, j \in \{1, \ldots, k\}$, $b \in \{0, 1\}$,

(c3) $x^i = y^i \vee u^j = v^j$ and $x^i = y^i \vee y^i = z^i$ for $i, j \in \{1, \ldots, k\}$, $i \neq j$,

(c4) $x^i + y^i = 1$ for $i \in \{1, \ldots, k\}$,

(c5) $x^i + y^i = u^j + v^j$ for $i, j \in \{1, \ldots, k\}$,

(c6)–(c7) $\cdots$

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(c1)–(c5) $\cdots$

(c6) $x^{s_1} + x^{s_2} + \cdots + x^{s_n} = b$, where $n \geq 2$, $s_1, \ldots, s_n \in \{1, \ldots, k\}$ are pairwise distinct integers, and $b \in \{0, 1\}$,

(c7)

$$\left\| \begin{array}{c} x_1^{s_1} = b_1 \vee x_2^{s_1} = b_1 \vee \cdots \vee x_{m_1}^{s_1} = b_1 \\ x_1^{s_2} = b_2 \vee x_2^{s_2} = b_2 \vee \cdots \vee x_{m_2}^{s_2} = b_2 \\ \vdots \\ x_1^{s_n} = b_n \vee x_2^{s_n} = b_n \vee \cdots \vee x_{m_n}^{s_n} = b_n \end{array} \right\|,$$

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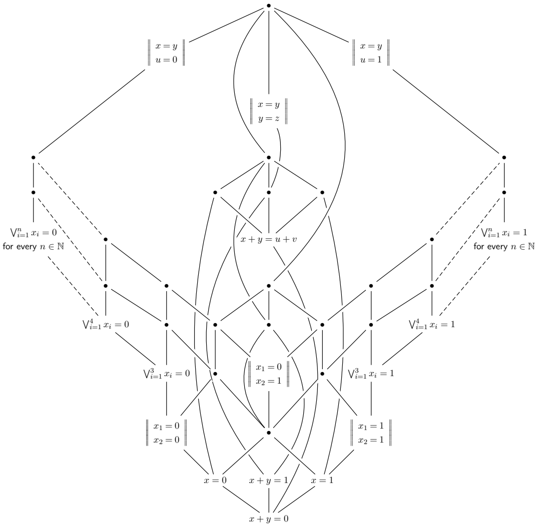
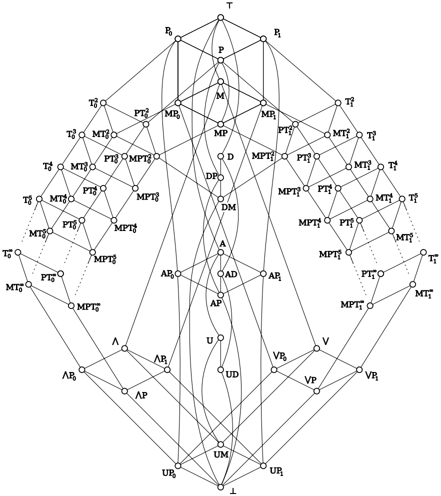
Post’s Lattice up to Non-surjective Operations





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example:

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Theorem

Let $k \in \mathbb{N}$. There is a finite poset $(F, \leq)$ such that the lattice of surjective k-clones embeds into the poset
$$(F, \geq) \times (\mathfrak{L}_{\downarrow}(\mathbb{N}_0^{2k}), \supseteq)\,.$$

Embedding of the Lattice of Clones

Let $\mathcal{C}$ be a k-sorted clone. We have the following decomposition:

$$\mathcal{C} = \{\boldsymbol{f} \in \mathcal{C} \mid \boldsymbol{f} \text{ is surjective}\} \cup \{\boldsymbol{f} \in \mathcal{C} \mid \boldsymbol{f} \text{ is not surjective}\}$$

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No infinite ascending chains:

Corollary (Taimanov)

Let $k \in \mathbb{N}$. Every k-clone on $\{0,1\}$ is generated by a finite set of operations.

Applications

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Idempotent clones
Clones of operations satisfying $f(x, \ldots, x) = x$ contain only surjective operations
$\to$ complete classification plausible – **current project**

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Description of a class of clones on $\{0,1,2\}$, ...

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Summary

Results

- Canonical relations
- Elementary operations & closedness criterion & simple proof of Post's lattice
- Embedding of the lattice of multi-sorted clones & simple proof of Taimanov's theorem

Promising framework

Everything looks "simple" in the presented perspective

Potential applications

Idempotent clones, CSPs, clones on different domains, ...

V. David, D. Zhuk, On the lattice of multi-sorted relational clones on a two-element set, arXiv:2505.06033 [math.LO] (preprint)